How to (Re)Invent Synthetic Tait Computability

PLunch
February 19, 2025

Runming Li

Metatheory for programming languages

- **Canonicity**
- Normalization
- Parametricity
- . . .

Metatheory for programming languages

- **Canonicity**
- Normalization
- Parametricity
- ...

Theorem (Canonicity)

Every closed term of type bool *is (or evaluates to) either* true *or* false.

Tait’s Computability Method: Logical Relations

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- Step 1: Define a **computability** predicate $[\![A]\!]$ by induction on types.
  - $M \in [\![\mathsf{bool}]\!]$ if $M = \mathsf{yes}$ or $M = \mathsf{no}$.

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  - $M \in [\![A \to B]\!]$ if for all $N \in [\![A]\!]$, $M\,N \in [\![B]\!]$.
  - $M \in [\![A \times B]\!]$ if $\pi_1 M \in [\![A]\!]$ and $\pi_2 M \in [\![B]\!]$.

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  - $M \in [\![A \times B]\!]$ if $\pi_1 M \in [\![A]\!]$ and $\pi_2 M \in [\![B]\!]$.
- Step 2: Prove that all well-typed terms are **computable**.

See Bob's 15-413/713 lecture notes for more details.

Let’s do it differently today!

I will use category theory . . .

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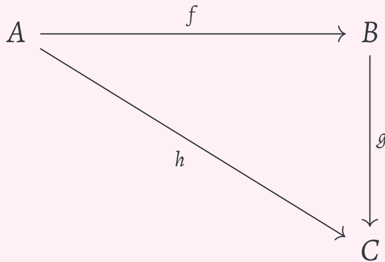
- A **category** is a collection of **objects** and **morphisms** between objects.



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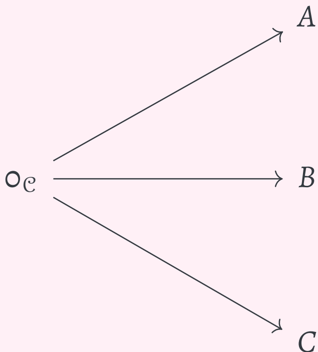
- An **initial object** is an object that has a unique morphism to every other object in the category.



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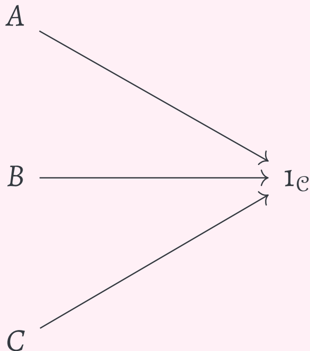
- A **terminal object** is an object that has a unique morphism from every other object in the category, usually written as $1_{\mathcal{C}}$.



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- A **functor** is a “function” between two categories.
- A $\mathrm{Hom}(A, B)$ is the set of morphisms from object A to object B.

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- A Π type is a dependent function type (think “for all” quantifier):

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Example

$$\Pi_{n:\mathbb{N}}\mathsf{even}(n) + \mathsf{odd}(n)$$

read as “for all n of type $\mathbb{N}$, n is either even or odd.”

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Example

$$\Sigma_{n:\mathbb{N}}(n = 42)$$

read as “there exists an n of type $\mathbb{N}$ such that $n = 42$.”

A motivating example

Theorem

Every natural number is either even or odd, i.e., a term of type $\Pi_{n:\mathbb{N}} \mathsf{even}(n) + \mathsf{odd}(n)$.

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Proof

By induction on n.

A categorical proof

Construct the following category $\mathcal{C}$:

- Objects: terms of type $\Sigma_{X:\mathsf{Type}}(1 + X \to X)$
  i.e., a pair of $(X, f : 1 + X \to X)$.

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  i.e., a pair of $(X, f : 1 + X \to X)$.
- Morphisms (between objects (X, f) and (Y, g)): a function $h : X \to Y$.

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$$f(\mathsf{inr}\,x) = \mathsf{succ}\,x$$

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$$Y : \mathsf{Type}$$
$$Y = \Sigma n : \mathbb{N}(\mathsf{even}(n) + \mathsf{odd}(n))$$

$$g : 1 + Y \to Y$$
$$g(\mathsf{inl}\,\star) = (\mathsf{zero}, \mathsf{zeroIsEven})$$
$$g(\mathsf{inr}\,(n, \mathsf{inl}\,p)) = (\mathsf{succ}\,n, \mathsf{inr}\,(\mathsf{evenOdd}(p)))$$
$$g(\mathsf{inr}\,(n, \mathsf{inr}\,p)) = (\mathsf{succ}\,n, \mathsf{inl}\,(\mathsf{oddEven}(p)))$$

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- There is a morphism from (X, f) to (Y, g): a function $\iota : \mathbb{N} \to \Sigma n : \mathbb{N}(\mathsf{even}(n) + \mathsf{odd}(n))$

What does ι look like?

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But maybe it is some random function that doesn't make sense?
e.g., $\iota(n) = (42, \mathsf{42\text{-}isEven})$?

Fundamental theorem

$$\iota : \mathbb{N} \to \Sigma n : \mathbb{N}(\mathsf{even}(n) + \mathsf{odd}(n))$$

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Our proof of **canonicity** would look much like this!

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We present an **equational theory** of simply-typed lambda calculus with only booleans and functions as a signature $\mathbb{SIG}$.

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  field
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    bool : tp
    yes : tm bool
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$\mathsf{tm} : \mathsf{tp} \to \textbf{Type}$

$\mathsf{bool} : \mathsf{tp}$

$\mathsf{yes} : \mathsf{tm}\ \mathsf{bool}$

$\mathsf{no} : \mathsf{tm}\ \mathsf{bool}$

$\mathsf{arr} : \mathsf{tp} \to \mathsf{tp} \to \mathsf{tp}$

$\mathsf{lam} : (\mathsf{tm}(A) \to \mathsf{tm}(B)) \to \mathsf{tm}\ (\mathsf{arr}\ A\ B)$

$\mathsf{app} : \mathsf{tm}\ (\mathsf{arr}\ A\ B) \to \mathsf{tm}\ A \to \mathsf{tm}\ B$

$\mathsf{arr}_\beta : \mathsf{app}\ (\mathsf{lam}\ f)\ x = f\ x$

$\mathsf{arr}_\eta : \mathsf{lam}\ (\mathsf{app}\ f) = f$

Example terms

Example

$$\mathsf{lam}\ (\lambda x.x) : \mathsf{tm}\ (\mathsf{arr}\ \mathsf{bool}\ \mathsf{bool})$$

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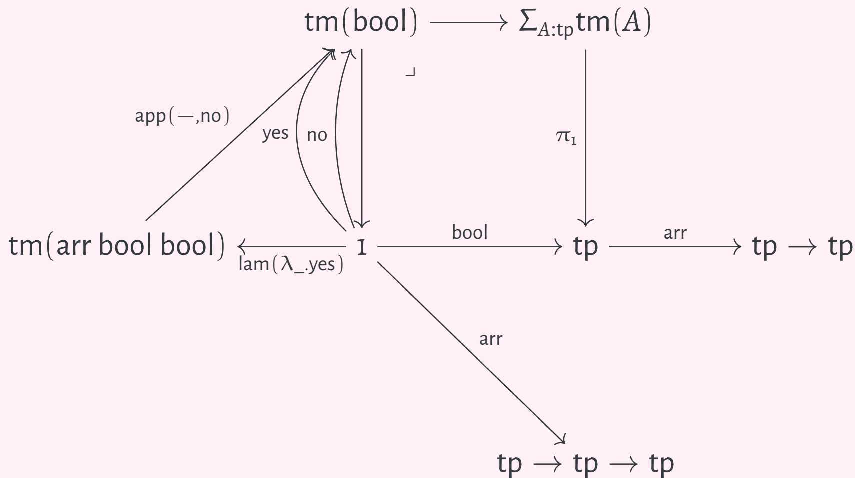
which is traditionally written as $(\lambda_.\mathsf{yes})\ \mathsf{no} : \mathsf{bool}$.
By using arr_β, we can show that the above term is equal to yes.

$\mathbb{SIG}$ induces a category $\mathcal{C}$



Some special morphisms

- Closed terms of type A are morphisms from $\mathbf{1}$ to $\mathsf{tm}(A)$.

$$\mathsf{yes} : \mathbf{1} \to \mathsf{tm}(\mathsf{bool})$$
$$\mathsf{no} : \mathbf{1} \to \mathsf{tm}(\mathsf{bool})$$
$$\mathsf{app}(-, \mathsf{no}) \circ \mathsf{lam}(\lambda_.\mathsf{yes}) : \mathbf{1} \to \mathsf{tm}(\mathsf{bool})$$

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Theorem (Canonicity)

For any morphism $b : \mathbf{1} \to \mathsf{tm}(\mathsf{bool})$, it must be the case that $b = \mathsf{yes}$ or $b = \mathsf{no}$.

Category of Computability Structures

Construct a category $\mathcal{E}$ as follows:

- Objects: computability structures $(A \in \mathcal{C}, S \in \mathbf{Set}, f : S \to \mathsf{Hom}_{\mathcal{C}}(1, A))$.

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- Morphisms: a morphism $b : A \to A'$ and a function $h : S \to S'$ such that:

$$
\begin{array}{ccc}
S & \xrightarrow{h} & S' \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\
\mathrm{Hom}(1, A) & \xrightarrow{\mathrm{Hom}(1,b)} & \mathrm{Hom}(1, A')
\end{array}
$$

What are some of the objects in $\mathcal{E}$?

- $\mathsf{tm}(\mathsf{bool}) = (\mathsf{tm}(\mathsf{bool}), \{\spadesuit, \clubsuit\}, f)$ where:
  $f(\spadesuit) = \mathsf{yes}$
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Define a functor $F : \mathcal{C} \to \mathcal{E}$ such that (in particular) $F(\mathsf{tm}(\mathsf{bool})) = \mathsf{tm}(\mathsf{bool})$.

Fundamental Theorem of Logical Relations

- Objects in $\mathcal{C}$: A.
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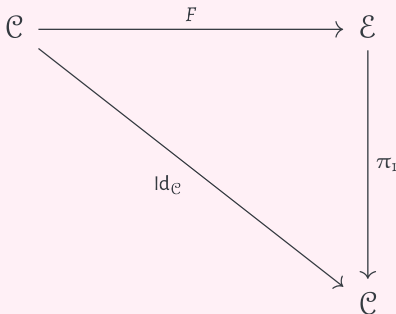

By construction, it must be the case that $\pi_1 \circ F = \mathsf{Id}_{\mathcal{C}}$.

Canonicity

Proof

Suppose we have a morphism $b : \mathbf{1} \to \mathsf{tm}(\mathsf{bool})$ in $\mathcal{C}$.

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Compute $F(b) : 1 \to \mathsf{tm}(\mathsf{bool})$.

Canonicity

Proof

Suppose we have a morphism $b : \mathbf{1} \to \mathsf{tm}(\mathsf{bool})$ in $\mathcal{C}$.
Compute $F(b) : \mathbf{1} \to \mathsf{tm}(\mathsf{bool})$.
This means that $F(b)$ consists of a morphism $b' : \mathbf{1} \to \mathsf{tm}(\mathsf{bool})$ and a function $h : \mathbf{1}_{\mathbf{Set}} \to \{\spadesuit, \clubsuit\}$ such that:

$$\begin{array}{ccc} \mathbf{1}_{\mathbf{Set}} & \xrightarrow{h} & \{\spadesuit, \clubsuit\} \\ \downarrow{!} & & \downarrow{f} \\ \mathbf{1}_{\mathbf{Set}} = \mathrm{Hom}(\mathbf{1}, \mathbf{1}) & \xrightarrow{b'} & \mathrm{Hom}(\mathbf{1}, \mathsf{tm}(\mathsf{bool})) \end{array}$$

Canonicity

Proof (Cont.)

Moreover, $\pi_1(F(b)) = b'$. By the fundamental theorem, $b' = b$.

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$$\begin{array}{ccc} 1_{\mathbf{Set}} & \xrightarrow{h} & \{\spadesuit, \clubsuit\} \\ \downarrow{!} & & \downarrow{f} \\ 1_{\mathbf{Set}} = \mathsf{Hom}(1, 1) & \xrightarrow{b} & \mathsf{Hom}(1, \mathsf{tm}(\mathsf{bool})) \end{array}$$

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Proof (Cont.)

Moreover, $\pi_1(F(b)) = b'$. By the fundamental theorem, $b' = b$.

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If $h = \spadesuit$, then $b = \mathsf{yes}$ (because $f(\spadesuit) = \mathsf{yes}$).
If $h = \clubsuit$, then $b = \mathsf{no}$ (because $f(\clubsuit) = \mathsf{no}$).

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- A construction of a computability structure category $\mathcal{E}$ by gluing syntax and semantics.
  - Follows a general construction of **Artin Gluing**.
- A construction of a functor $F : \mathcal{C} \to \mathcal{E}$.
  - Tedious! A lot of conditions to check.
  - F is a **functorial model** of the language.

What is a model?

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Any implementation $\mathsf{M} : \mathbb{SIG}$ is a **model** of the language!

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$$\begin{aligned}
\mathsf{M.tp} &= \mathbf{Type} \\
\mathsf{M.tm}(A) &= A \\
\\
\mathsf{M.bool} &= 1 + 1 \\
\mathsf{M.yes} &= \mathsf{inl}\,\star \\
\mathsf{M.no} &= \mathsf{inr}\,\star \\
\\
\mathsf{M.arr}\,A\,B &= A \to B \\
\mathsf{M.lam}\,f &= f \\
\mathsf{M.app}\,f\,x &= f\,x
\end{aligned}$$

Our Plan

- Suppose we have a model $\mathsf{M} : \mathbb{SIG}$ that corresponds to the syntax.
- Construct a model $\mathsf{M} : \mathbb{SIG}$ that corresponds to the gluing of syntax and semantics, such that

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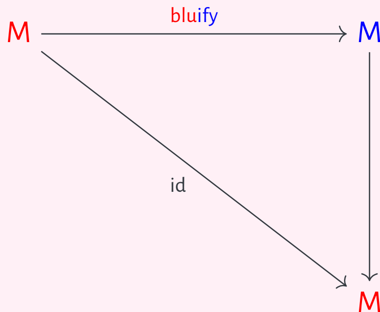
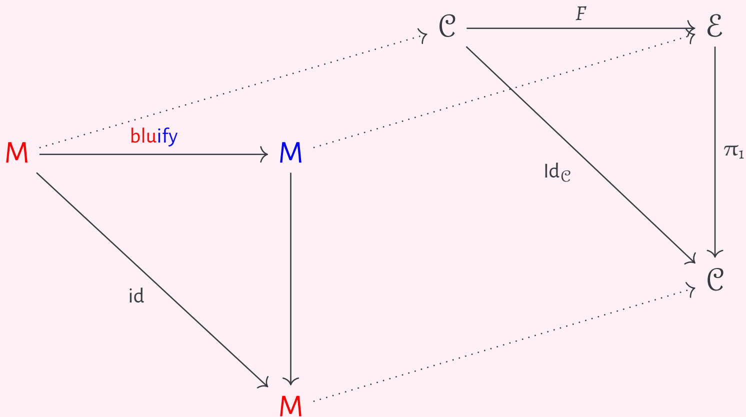


Piecing things together



Some machinary in the dependent type theory

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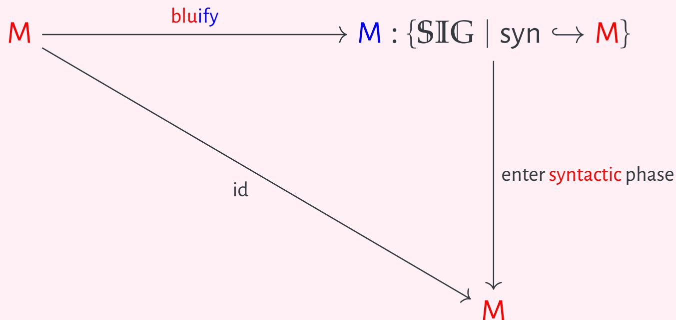
- A proposition $\mathsf{syn} : \mathsf{Prop}$.
  - If syn holds, then we say that we are in the **syntactic phase**.
- Extension type: $\{A \mid \mathsf{syn} \hookrightarrow a_\circ\}$ where $a_\circ : A$.
  - A term $a : \{A \mid \mathsf{syn} \hookrightarrow a_\circ\}$ is a term $a : A$ such that under the **syntactic phase** $a = a_\circ$.

New goal

Construct $\mathsf{M} : \{\mathbb{SIG} \mid \mathsf{syn} \hookrightarrow \mathsf{M}\}$.



Constructing M

$$\mathsf{M.tp} : \{\mathbf{Type} \mid \mathsf{syn} \hookrightarrow \mathsf{M.tp}\}$$
$$\mathsf{M.tm} : \{\mathsf{M.tp} \to \mathbf{Type} \mid \mathsf{syn} \hookrightarrow \mathsf{M.tm}\}$$

$$\mathsf{M.bool} : \{\mathsf{M.tp} \mid \mathsf{syn} \hookrightarrow \mathsf{M.bool}\}$$
$$\mathsf{M.yes} : \{\mathsf{M.tm}(\mathsf{M.bool}) \mid \mathsf{syn} \hookrightarrow \mathsf{M.yes}\}$$
$$\mathsf{M.no} : \{\mathsf{M.tm}(\mathsf{M.bool}) \mid \mathsf{syn} \hookrightarrow \mathsf{M.no}\}$$

$$\ldots$$

Constructing M

$$\mathsf{M.tp} : \{\mathbf{Type} \mid \mathsf{syn} \hookrightarrow \mathsf{M.tp}\}$$

$$\mathsf{M.tp} = \Sigma_{A:\mathsf{M.tp}}\{\mathbf{Type} \mid \mathsf{syn} \hookrightarrow \mathsf{M.tm}(A)\}$$

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Think as: the computability structure of $\mathsf{M.tp}$ is for each syntactic type A, a collection of terms of that type and proofs that those terms are computable.

Constructing M

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Check: under the syntactic phase (assuming syn),

$$\begin{aligned} &\mathsf{M.tp} \\ =&\Sigma_{A:\mathsf{M.tp}}\{\mathbf{Type} \mid \mathsf{syn} \hookrightarrow \mathsf{M.tm}(A)\} \\ \cong&\Sigma_{A:\mathsf{M.tp}}\mathbf{1} \\ \cong&\mathsf{M.tp} \end{aligned}$$

Constructing M

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$$\mathsf{M.tm} : \{\mathsf{M.tp} \to \mathbf{Type} \mid \mathsf{syn} \hookrightarrow \mathsf{M.tm}\}$$

$$\mathsf{M.tm}(A) = \pi_2 A$$

Constructing M

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$$\mathsf{M.bool} : \{\mathsf{M.tp} \mid \mathsf{syn} \hookrightarrow \mathsf{M.bool}\}$$

$$\mathsf{M.bool} = (\mathsf{M.bool}, \Sigma_{b:\mathsf{tm}(\mathsf{bool})}(b = \mathsf{M.yes}) + (b = \mathsf{M.no}))$$ [1]

$$\mathsf{M.yes} : \{\mathsf{M.tm}(\mathsf{M.bool}) \mid \mathsf{syn} \hookrightarrow \mathsf{M.yes}\}$$

$$\mathsf{M.yes} = (\mathsf{M.yes}, \mathsf{inl}(\checkmark))$$

[1] Well, I lied slightly.

Constructing M

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In almost all cases, there is only one way that makes the type-checker happy.

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In almost all cases, there is only one way that makes the type-checker happy.

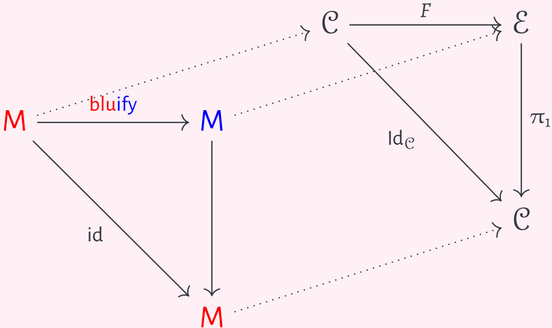
Just like in traditional Logical Relations, there is no creativity beyond the base types.

And that is Synthetic Tait's Computability!

What is this proof?

What is this proof?

- A programming exercise to construct $\mathsf{M} : \{\mathbb{SIG} \mid \mathsf{syn} \hookrightarrow \mathsf{M}\}$ in a dependently typed language.
- Everything else can be black-boxed if you don't want to deal with category theory.



Synthetic Tait Computability in Real World

Parametricity for an ML module calculus	Sterling & Harper
Normalization for Cartesian Cubical Type Theory	Sterling & Angiuli
Normalization for a multimodal type theory	Gratzer
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Canonicity for Cost-Aware Logical Framework	

Conclusion

- Syntax and semantics of a programming language displays a **phase distinction** that can be manipulated synthetically.
- **Synthetic Tait Computability** exploits this by **gluing** syntax and semantics together.
- Proving meta-theoretic properties by Logical Relations can be reduced to a programming exercise in a dependently typed language.